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Isometric Energies for Recovering Injectivity in Constrained Mapping

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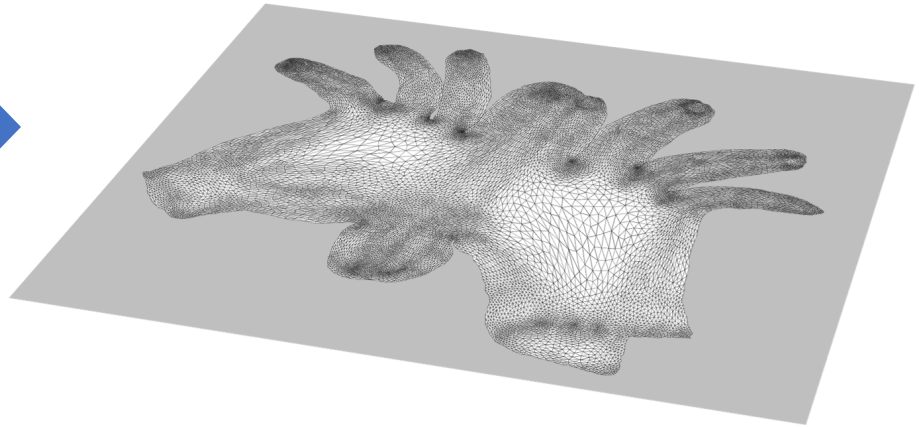
Mapping Meshes to another domain

triangle mesh

map

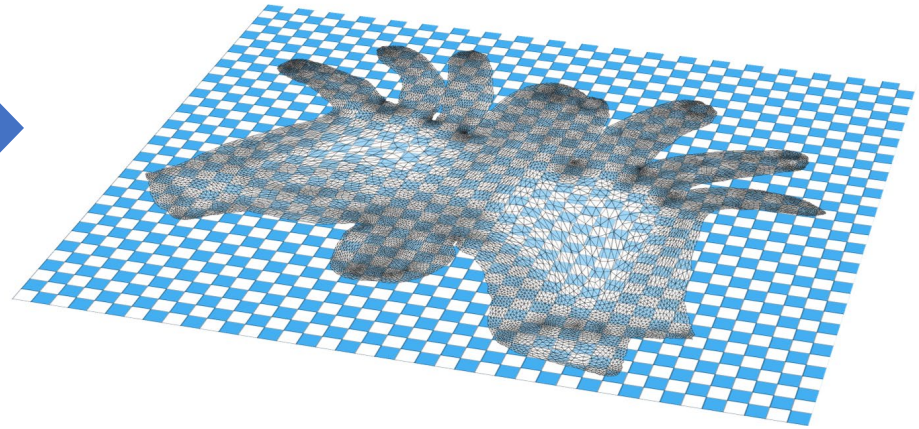


2D plane

Application: Texture mapping

triangle mesh

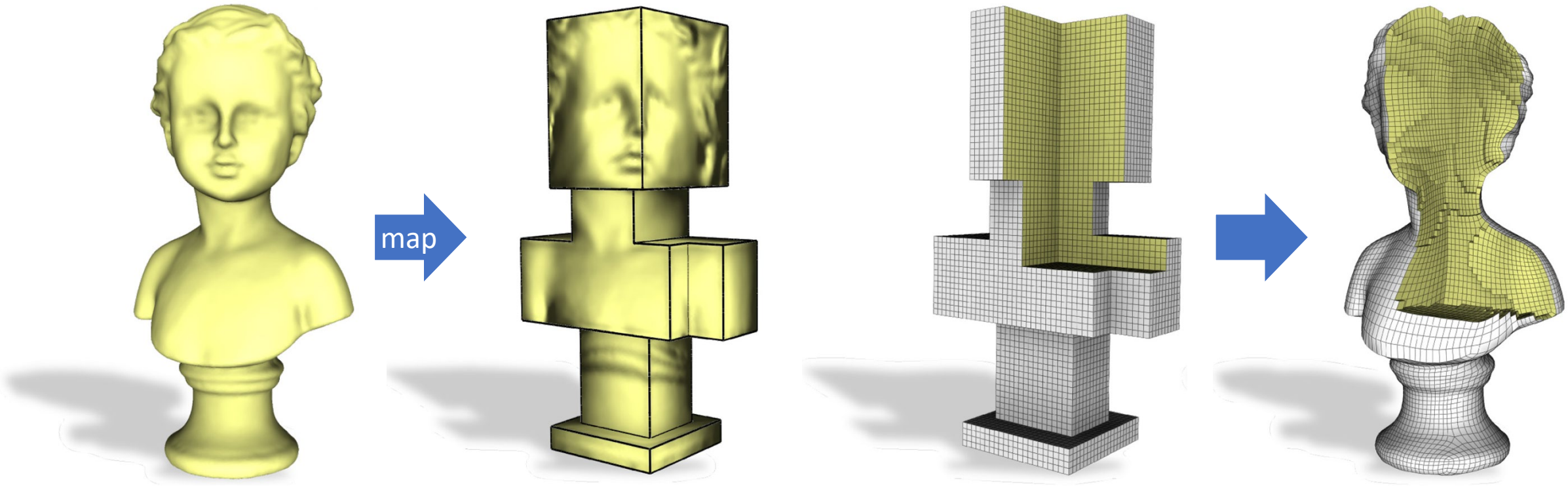
map



2D plane

textured mesh

Application: Hex-meshing



tetrahedron mesh

3D domain

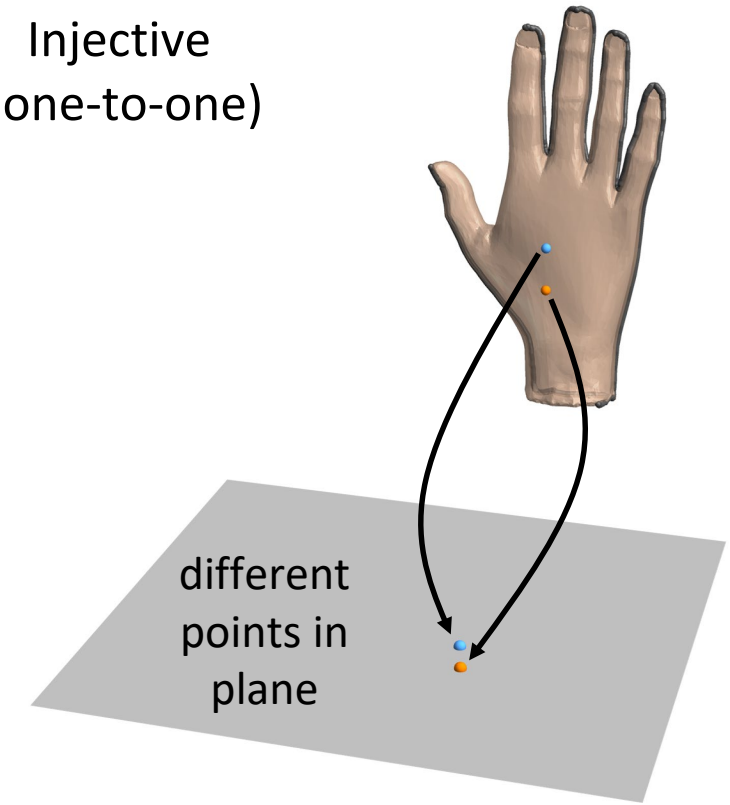
regular grid

hex-mesh

[Nico et al. 2022]

Injective (one-to-one) Mapping

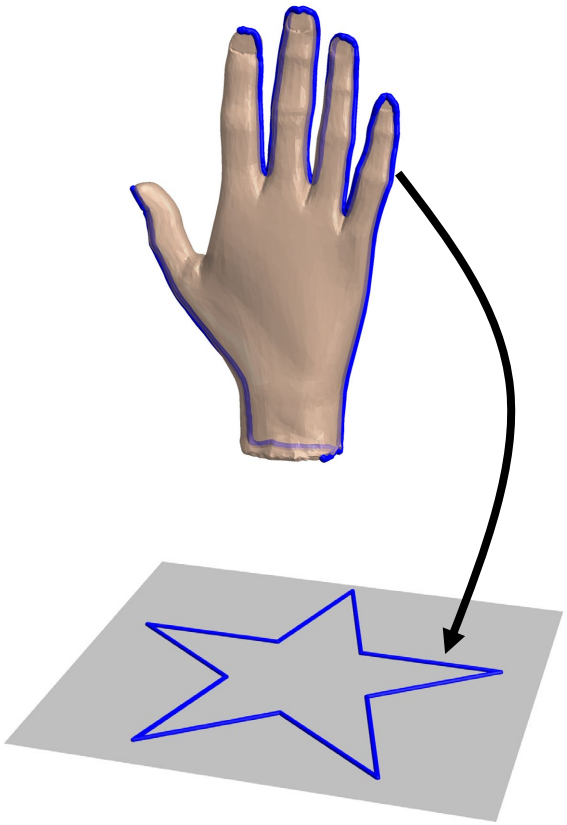
Injective
(one-to-one)



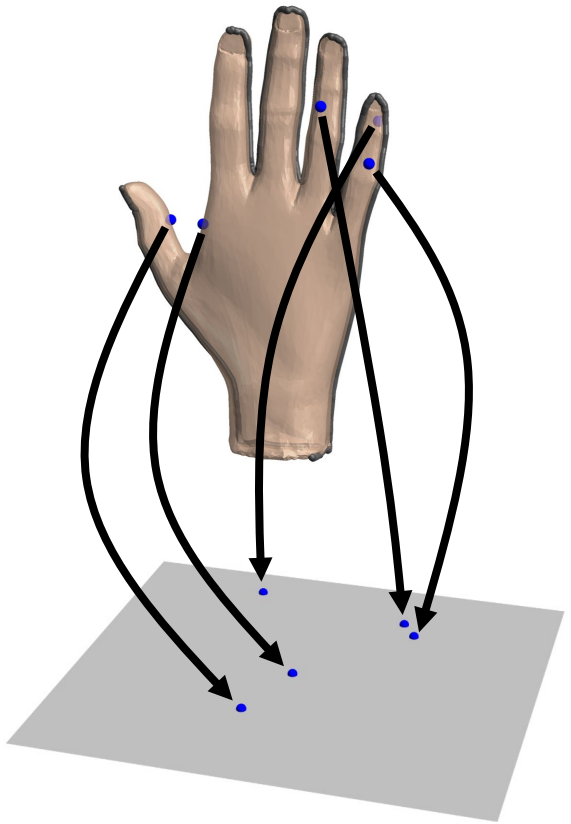
Non-injective

unusable
for many
applications

Constrained mapping

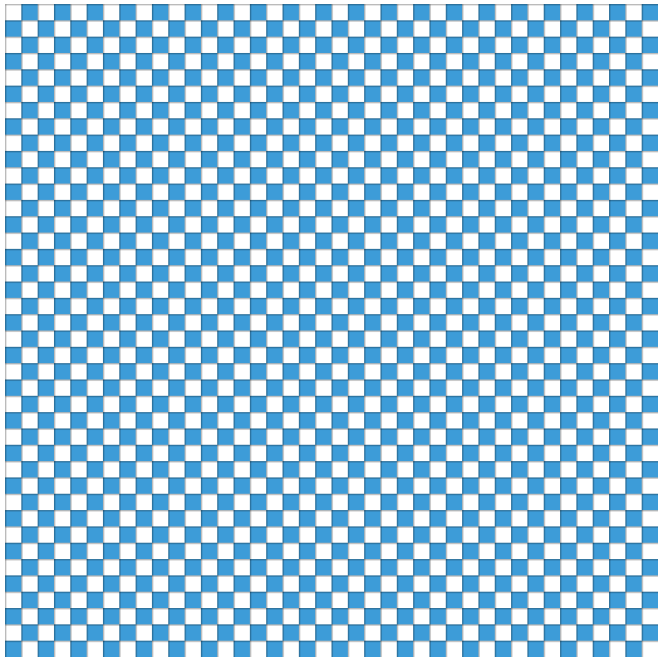


mapping into
prescribed boundary



mapping with
positional constraints

Mapping distortions



texture image

high-quality map

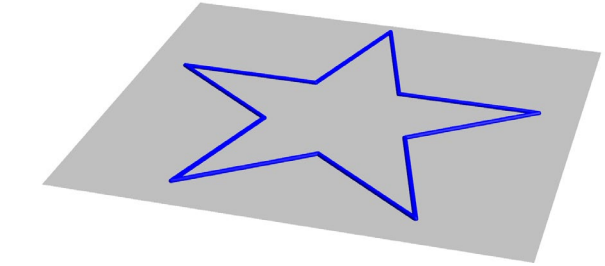
low-quality map

large distortion

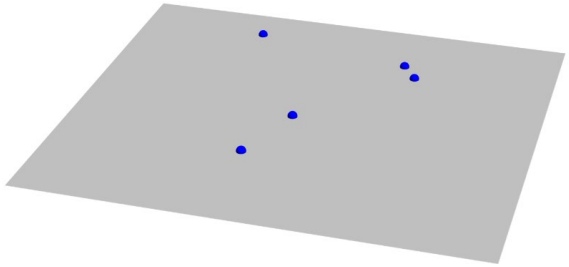
Goal: Injective Low-Distortion Map with Constraints

Input

(1) source mesh

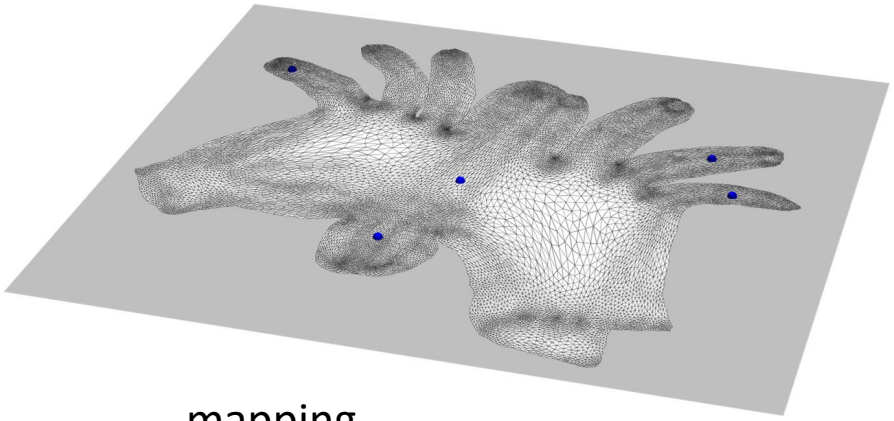


or



(2) prescribed boundary or positional constraints

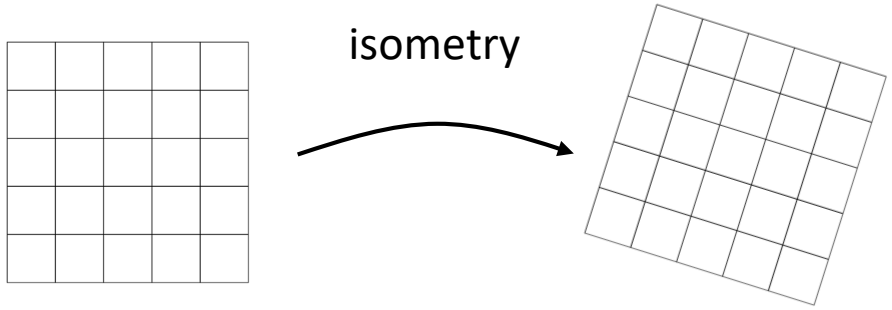
Output

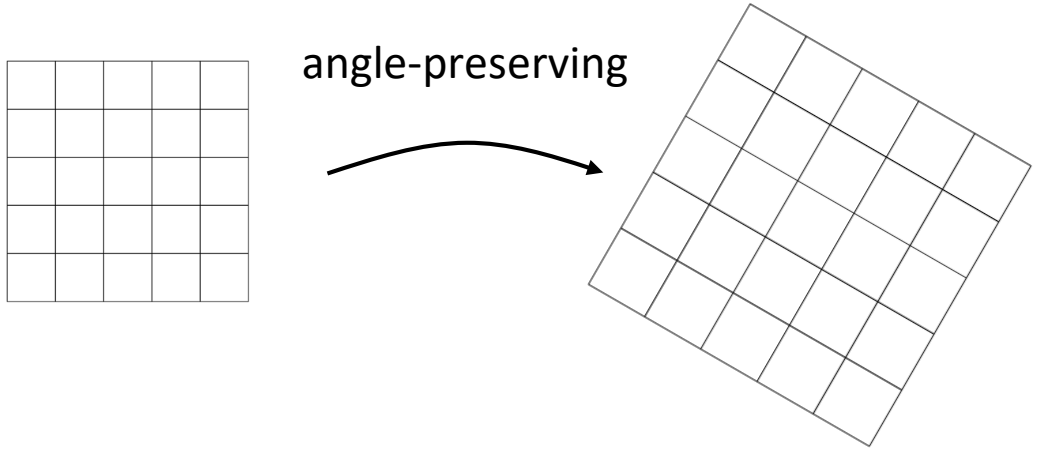


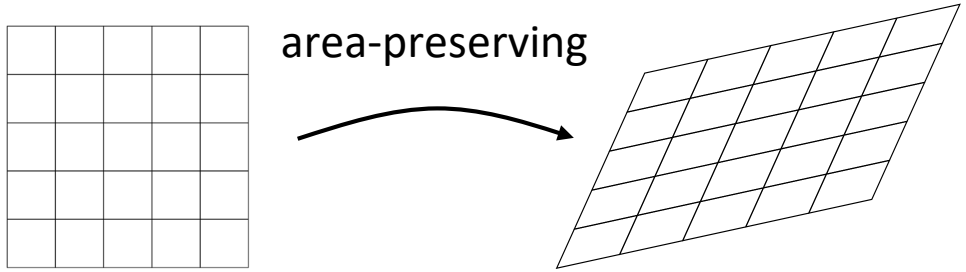
mapping

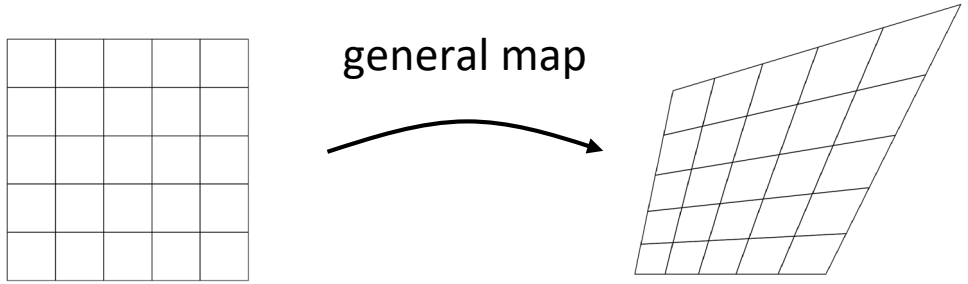
- constraint-satisfying
- injective
- low-distortion

Mapping distortions





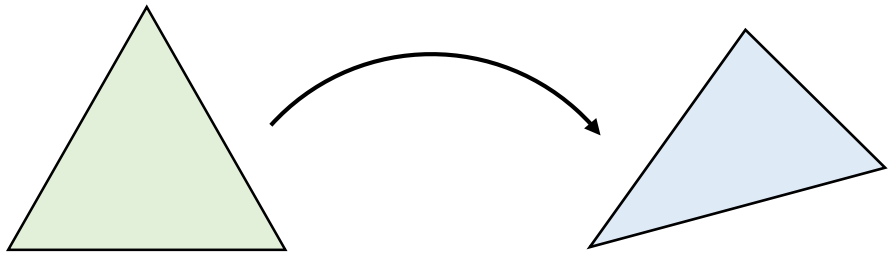




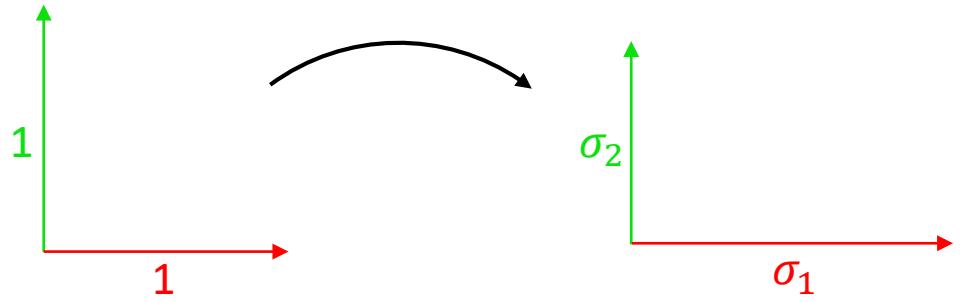
Mapping distortion

source triangle

target triangle

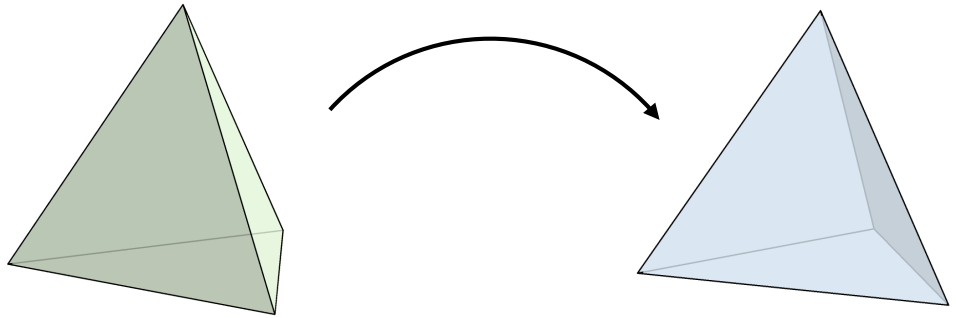


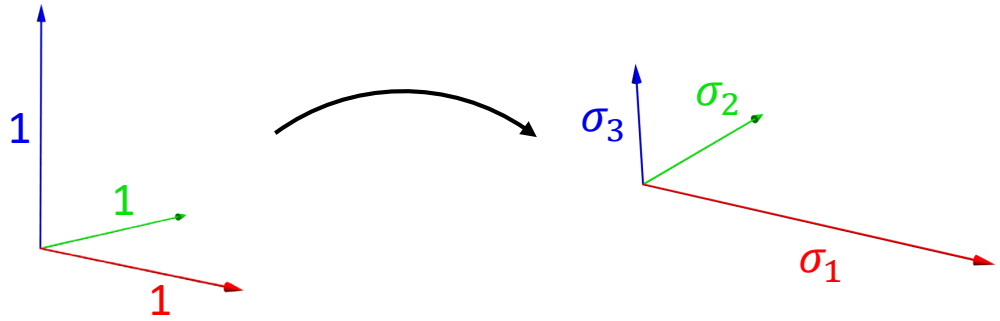
singular values



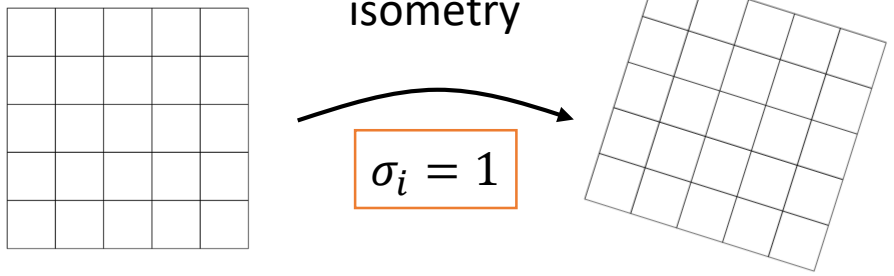
source tetrahedron

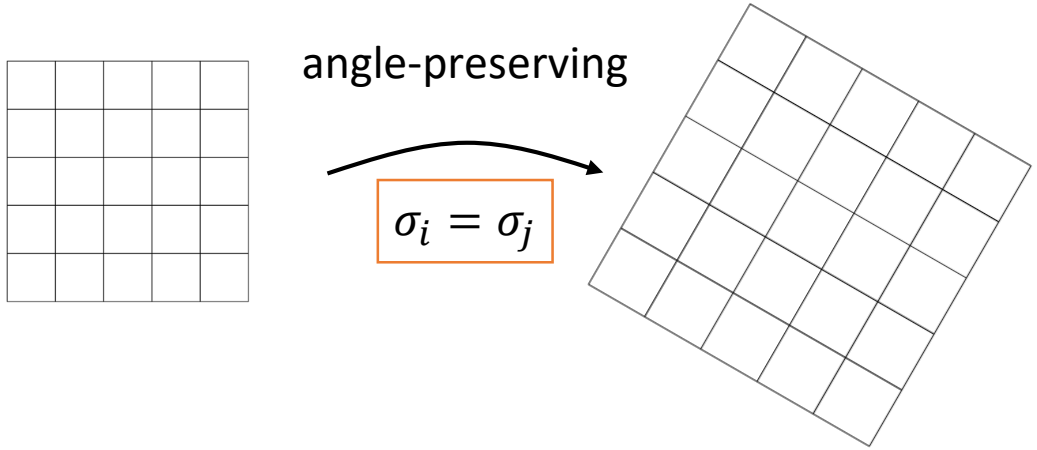
target tetrahedron

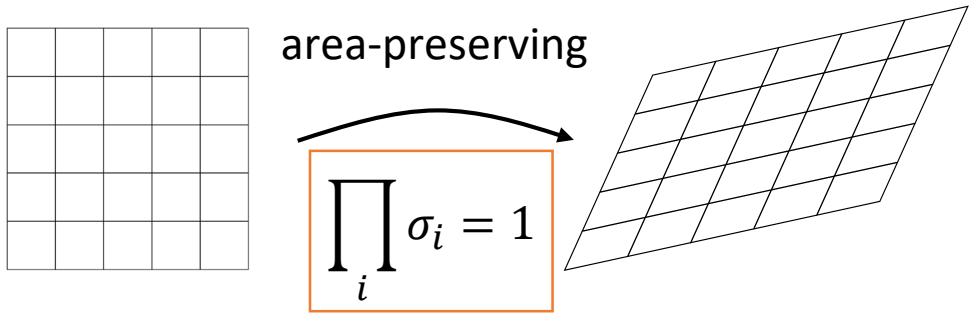




Mapping distortions







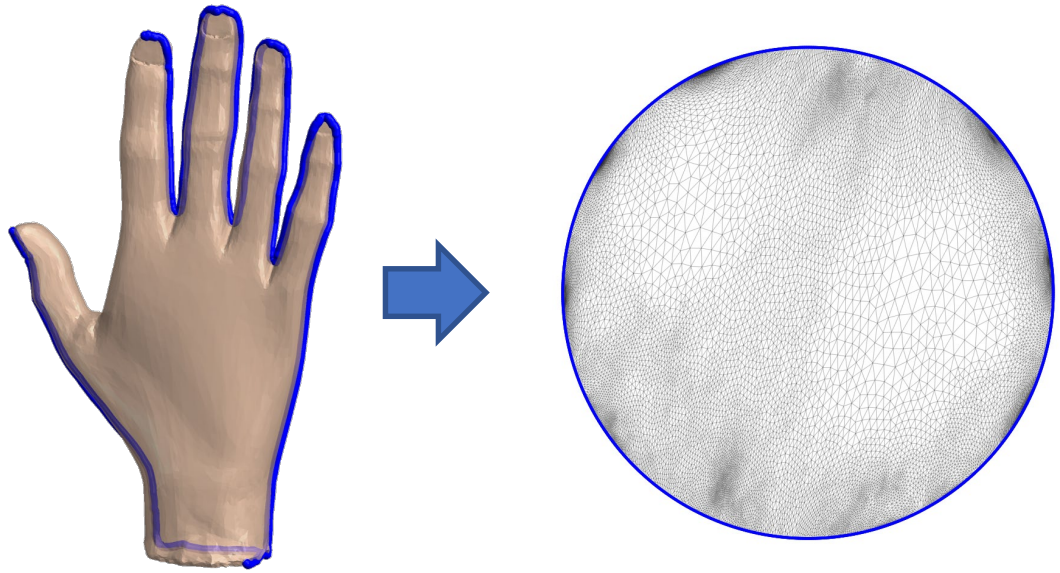
Isometric distortion metric

2D: $\max(\sigma_1, \frac{1}{\sigma_2})$ 3D: $\max(\sigma_1, \frac{1}{\sigma_3})$

minimized when $\sigma_i = 1$

Existing works

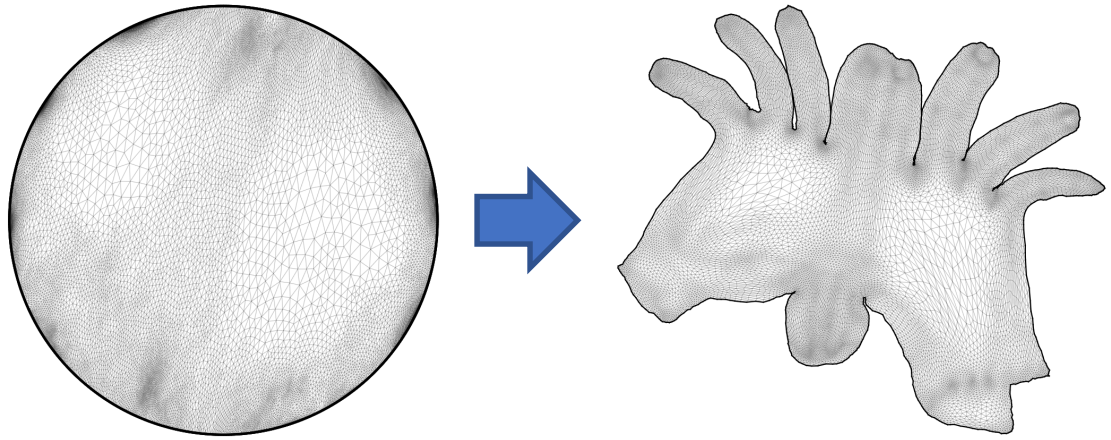
- Tutte embedding [Tutte 1963]
  - Convex boundary and no positional constraints



Tutte embedding
[Tutte 1963]

Existing works

- Tutte embedding [Tutte 1963]
  - Convex boundary and no positional constraints
- Maintenance methods
  - [Hormann and Greiner 2000], [Schüller et al. 2013], [Smith and Schaefer 2015], AMIPS [Fu et al. 2015], [Liu et al. 2016], SLIM [Rabinovich et al. 2017], CM [Shtengel et al. 2017], [Claici et al. 2017], SCAF[Jiang et al. 2017], BCQN [Zhu et al. 2018], [Liu et al. 2018], [Su et al. 2020], IDP [Fang et al. 2021]
  - Require injective maps to initialize

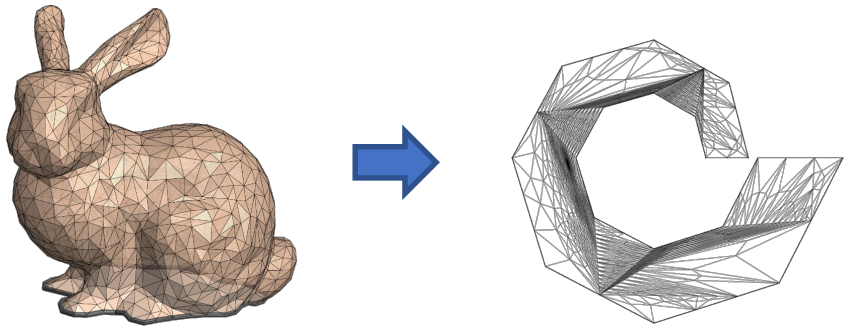


Injective initial map

SCAF
[Jiang et al. 2017]

Existing works

- Tutte embedding [Tutte 1963]
  - Convex boundary and no positional constraints
- Maintenance methods
  - [Hormann and Greiner 2000], [Schüller et al. 2013], [Smith and Schaefer 2015], AMIPS [Fu et al. 2015], [Liu et al. 2016], SLIM [Rabinovich et al. 2017], CM [Shtengel et al. 2017], [Claici et al. 2017], SCAF[Jiang et al. 2017], BCQN [Zhu et al. 2018], [Liu et al. 2018], [Su et al. 2020], IDP [Fang et al. 2021]
  - Require injective maps to initialize
- Remeshing
  - [Eckstein et al. 2001], Matchmaker [Kraevoy et al. 2003], [Lee et al. 2008], [Agarwal et al. 2008], [Weber and Zorin 2014], [Gu et al. 2018], [Shen et al. 2019]
  - May change the mesh structure



1024 vertices → 1743 vertices

Progressive Embedding
[Shen et al. 2019]

Existing works

- Tutte embedding [Tutte 1963]
  - Convex boundary and no positional constraints
- Maintenance methods
  - [Hormann and Greiner 2000], [Schüller et al. 2013], [Smith and Schaefer 2015], AMIPS [Fu et al. 2015], [Liu et al. 2016], SLIM [Rabinovich et al. 2017], CM [Shtengel et al. 2017], [Claici et al. 2017], SCAF[Jiang et al. 2017], BCQN [Zhu et al. 2018], [Liu et al. 2018], [Su et al. 2020], IDP [Fang et al. 2021]
  - Require injective maps to initialize
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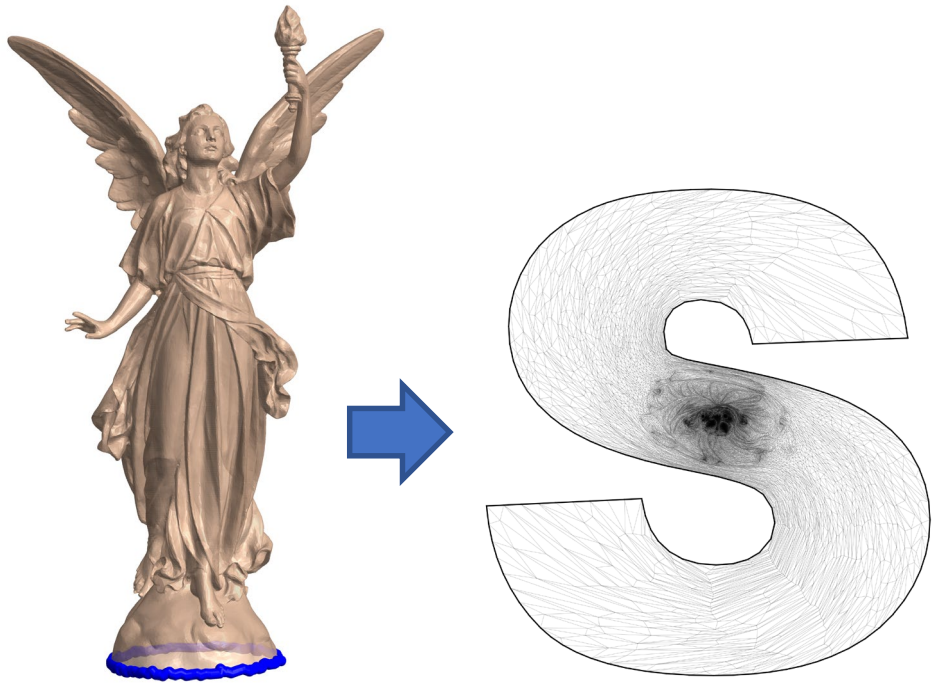
  - [Eckstein et al. 2001], Matchmaker [Kraevoy et al. 2003], [Lee et al. 2008], [Agarwal et al. 2008], [Weber and Zorin 2014], [Gu et al. 2018], [Shen et al. 2019]
  - May change the mesh structure
- Injectivity recovery
  - [Aigerman and Lipman 2013], [Kovalsky et al. 2015], [Fu and Liu 2016], [Su et al. 2019], [Hefetz et al. 2019], [Du et al. 2020], [Garanza et al. 2021], [Du et al. 2021], [Garanza et al. 2022], [Wang et al. 2022]



FFM
[Garanzha et al. 2021]

Existing works

- Tutte embedding [Tutte 1963]
  - Convex boundary and no positional constraints
- Maintenance methods
  - [Hormann and Greiner 2000], [Schüller et al. 2013], [Smith and Schaefer 2015], AMIPS [Fu et al. 2015], [Liu et al. 2016], SLIM [Rabinovich et al. 2017], CM [Shtengel et al. 2017], [Claici et al. 2017], SCAF[Jiang et al. 2017], BCQN [Zhu et al. 2018], [Liu et al. 2018], [Su et al. 2020], IDP [Fang et al. 2021]
  - Require injective maps to initialize
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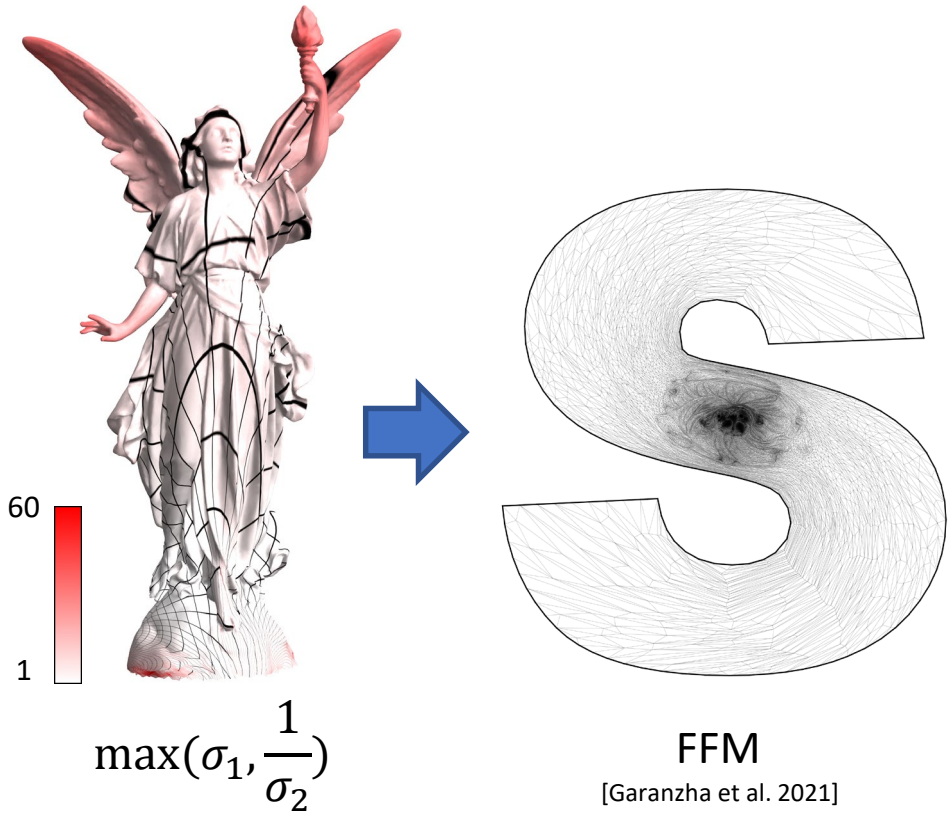
  - [Eckstein et al. 2001], Matchmaker [Kraevoy et al. 2003], [Lee et al. 2008], [Agarwal et al. 2008], [Weber and Zorin 2014], [Gu et al. 2018], [Shen et al. 2019]
  - May change the mesh structure
- Injectivity recovery
  - [Aigerman and Lipman 2013], [Kovalsky et al. 2015], [Fu and Liu 2016], [Su et al. 2019], [Hefetz et al. 2019], [Du et al. 2020], [Garanza et al. 2021], [Du et al. 2021], [Garanza et al. 2022], [Wang et al. 2022]
  - ~~Large area distortion~~ reduce area distortion

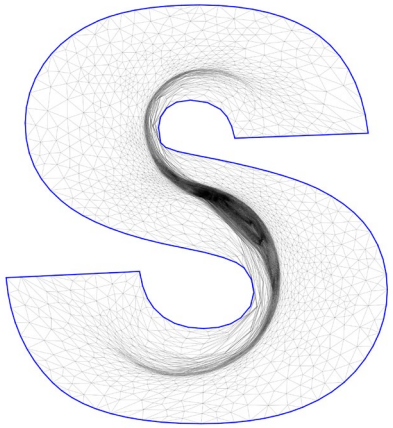


Contribution

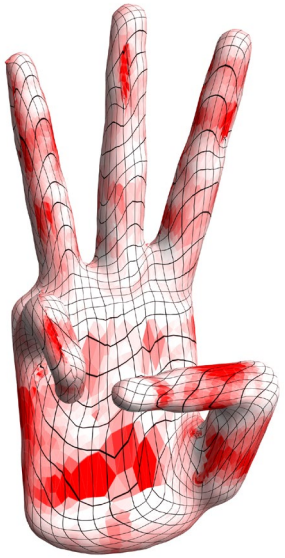
- Modification of existing energies

Fixed-boundary mapping

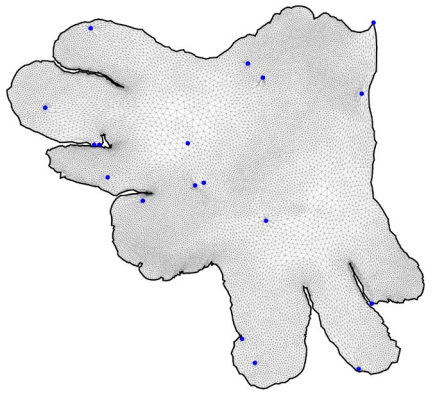
TLC
[Du et al. 2020]



Free-boundary mapping



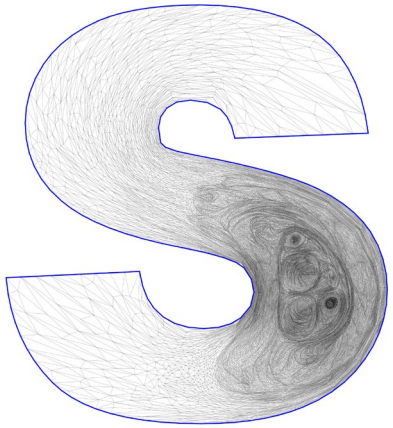
SEA
[Du et al. 2021]

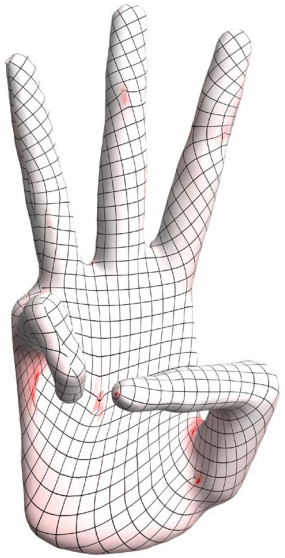


Contribution

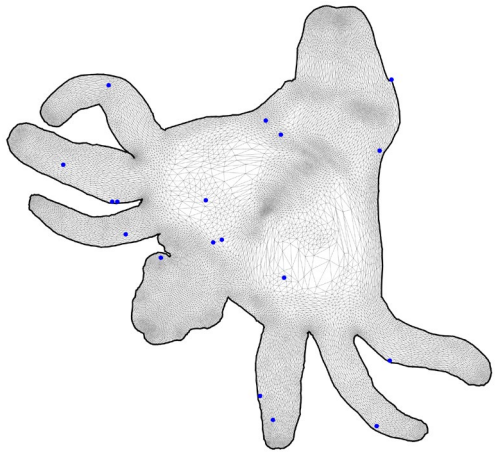
- Modification of existing energies
  - Maintain high success rate
  - Reduce area distortion

IsoTLC





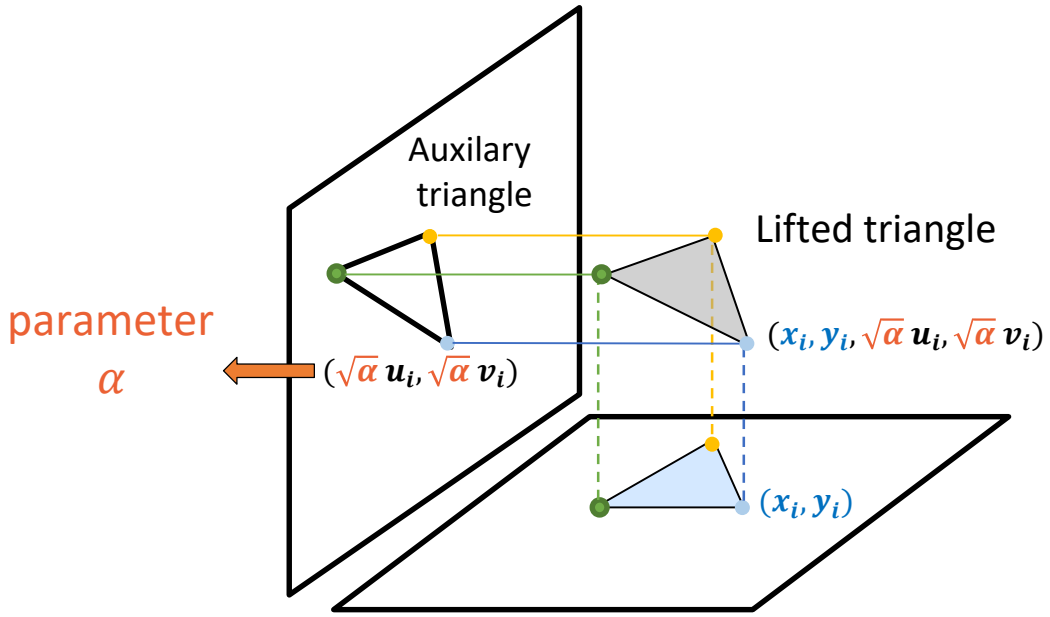
IsoSEA



Fixed-boundary mapping

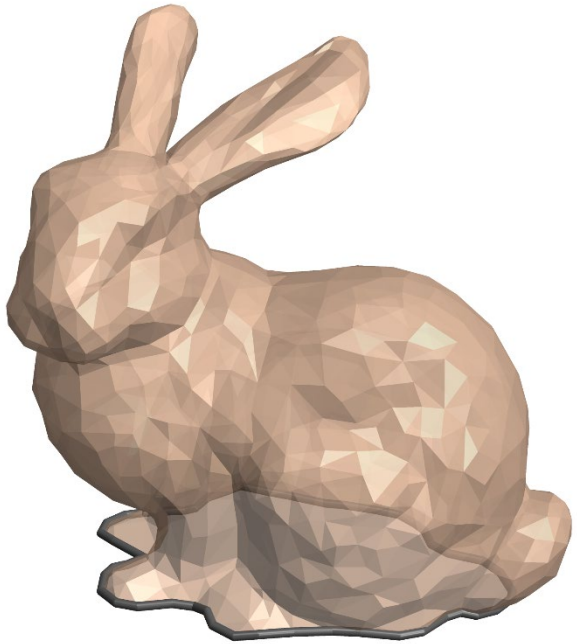
- Total lifted content (TLC) [Du et al. 2020]
  - Sum of area (volume) of lifted simplices

- Properties
  - Smoothly defined for injective and non-injective maps
  - Well-suited for gradient-based solvers
  - Global minima are injective maps (if they exist)
  - High success rate in recovering injectivity in 2D and 3D benchmarks

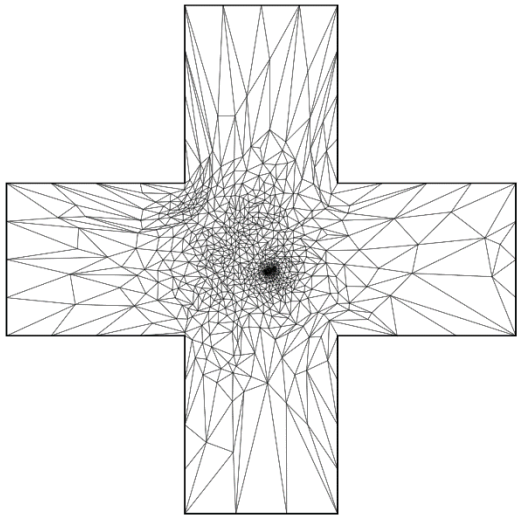


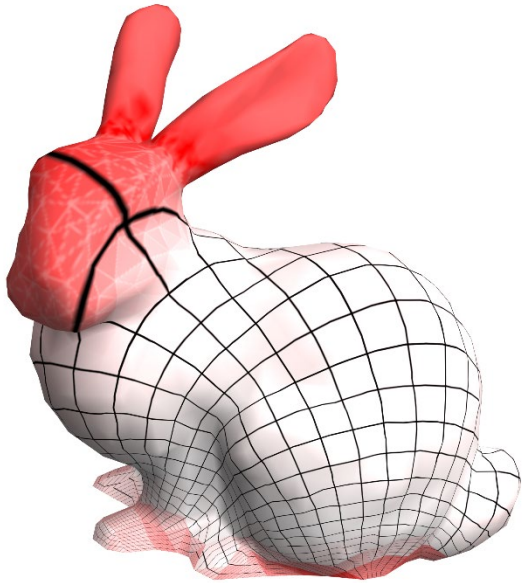
Fixed-boundary mapping

- TLC introduces area distortions



TLC





$$\max(\sigma_1, \frac{1}{\sigma_2})$$

20

1

TLC and distortion

- Auxiliary triangle $\tilde{t}$, triangle t

residual lifted area signed area

$$R_{\tilde{t},\alpha}(t) = A_{\tilde{t},\alpha}(t) - A(t)$$

same minimizer constant

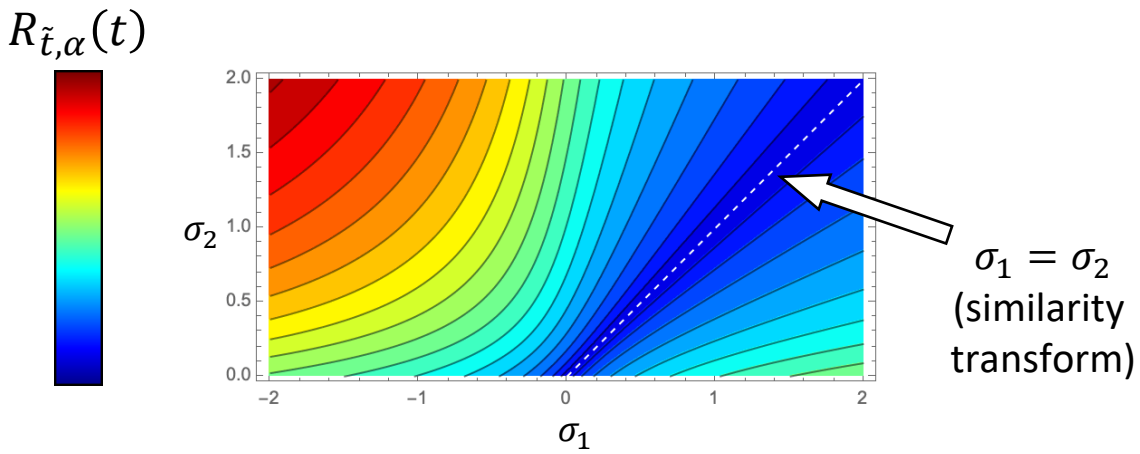
TLC and distortion

- Auxiliary triangle $\tilde{t}$, triangle t

residual aux area lifted area signed area

$$R_{\tilde{t},\alpha}(t) = A_{\tilde{t}}\left(\sqrt{(\sigma_1^2+\alpha)(\sigma_2^2+\alpha)} - \sigma_1\sigma_2\right)$$

- The residual is minimized when $\sigma_1 = \sigma_2 \geq 0$



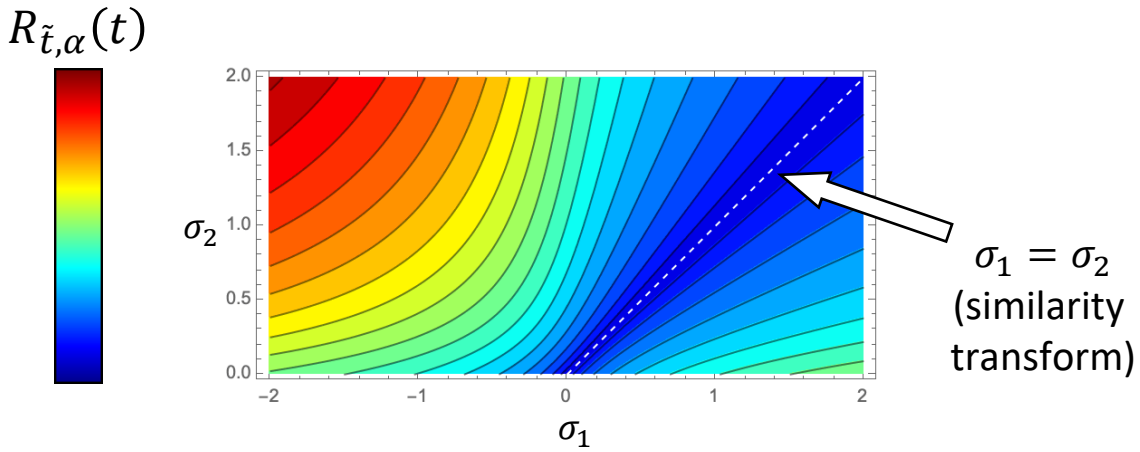
TLC and distortion

- Auxiliary triangle $\tilde{t}$, triangle t

lifted area

$$R_{\tilde{t},\alpha}(t) = A_{\tilde{t}}\left(\sqrt{(\sigma_1^2+\alpha)(\sigma_2^2+\alpha)} - \sigma_1\sigma_2\right)$$

- The residual is minimized when $\sigma_1 = \sigma_2 \geq 0$



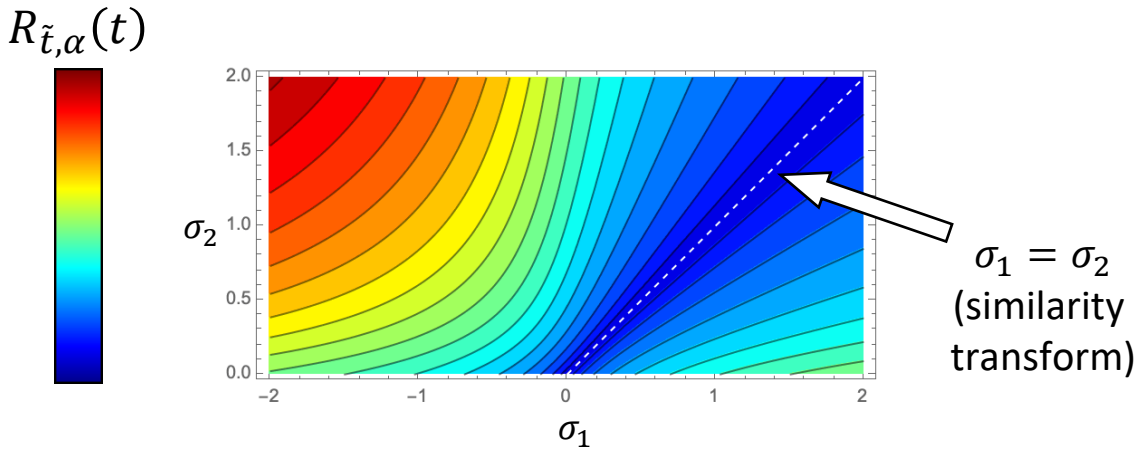
TLC and distortion

- Auxiliary triangle $\tilde{t}$, triangle t

MIPS
[Hormann and Greiner 2002]

$$R_{\tilde{t},\alpha}(t) = A_{\tilde{t}}\left(\sqrt{\sigma_1^2\sigma_2^2 + \alpha\sigma_1\sigma_2\left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right) + \alpha^2} - \sigma_1\sigma_2\right)$$

- The residual is minimized when $\sigma_1 = \sigma_2 \geq 0$



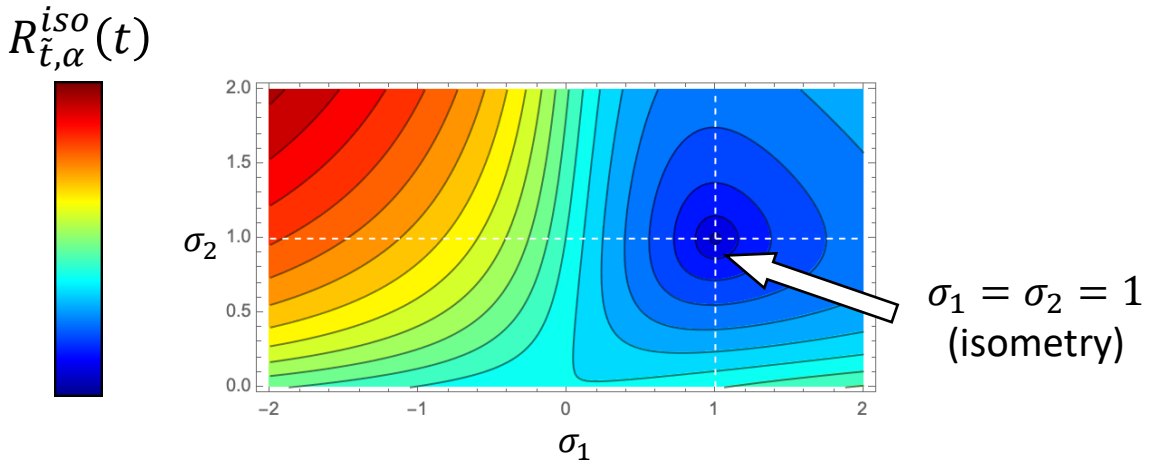
TLC and distortion

- Auxiliary triangle $\tilde{t}$, triangle t

isometric lifted content

$$R^{iso}_{\tilde{t},\alpha}(t) = A_{\tilde{t}}\left(\sqrt{\sigma_1^2\sigma_2^2 + \alpha\sigma_1\sigma_2\frac{1}{2}\left(\sigma_1 + \frac{1}{\sigma_1}\right)\left(\sigma_2 + \frac{1}{\sigma_2}\right) + \alpha^2} - \sigma_1\sigma_2\right)$$

- The residual is minimized when $\boldsymbol{\sigma_1 = \sigma_2 = 1}$



Isometric lifted content

- Auxiliary triangle $\tilde{t}$, **triangle** t

$$A^{iso}_{\tilde{t},\alpha}(t) = A_{\tilde{t}} \sqrt{\sigma_1^2\sigma_2^2 + \alpha\sigma_1\sigma_2 \frac{1}{2}\left(\sigma_1 + \frac{1}{\sigma_1}\right)\left(\sigma_2 + \frac{1}{\sigma_2}\right) + \alpha^2}$$

The residual is minimized when
$\boldsymbol{\sigma_1 = \sigma_2 = 1}$

- Auxiliary tetrahedron $\tilde{t}$, **tetrahedron** t

$$A^{iso}_{\tilde{t},\alpha}(t) = A_{\tilde{t}} \sqrt{\sigma_1^2\sigma_2^2\sigma_3^2 + \alpha\sigma_1\sigma_2\sigma_3 \frac{1}{4}\left(\sigma_1 + \frac{1}{\sigma_1}\right)\left(\sigma_2 + \frac{1}{\sigma_2}\right)\left(\sigma_3 + \frac{1}{\sigma_3}\right) + \alpha^2}$$

The residual is minimized when
$\boldsymbol{\sigma_1 = \sigma_2 = \sigma_3 = 1}$

Fixed-boundary mapping

- Isometric total lifted content (IsoTLC)
  - Sum of isometric lifted content

$$\mathrm{IsoTLC}(Mesh) = \sum_{\text{simplex } t \in Mesh} A^{iso}_{\tilde{t},\alpha}(t)$$

- Properties
  - Smoothly defined for injective and non-injective maps
  - Well-suited for gradient-based solvers
  - Global minima are injective maps (if they exist)
  - High success rate in recovering injectivity in 2D and 3D benchmarks
  - Low area distortion

Fixed-boundary mapping

- Isometric total lifted content (IsoTLC)
  - Sum of isometric lifted content
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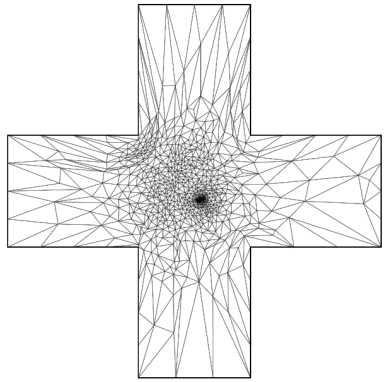
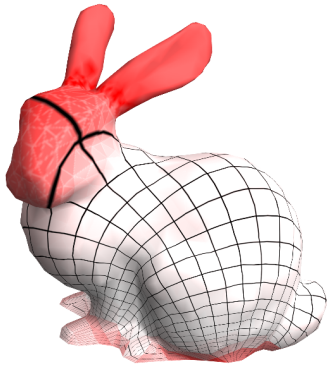
  - Low area distortion

TLC



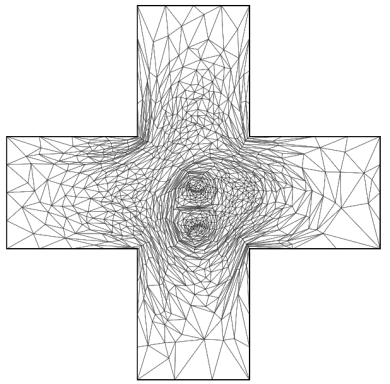


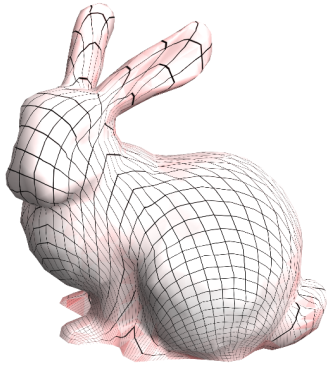
$\max(\sigma_1, \frac{1}{\sigma_2})$

20

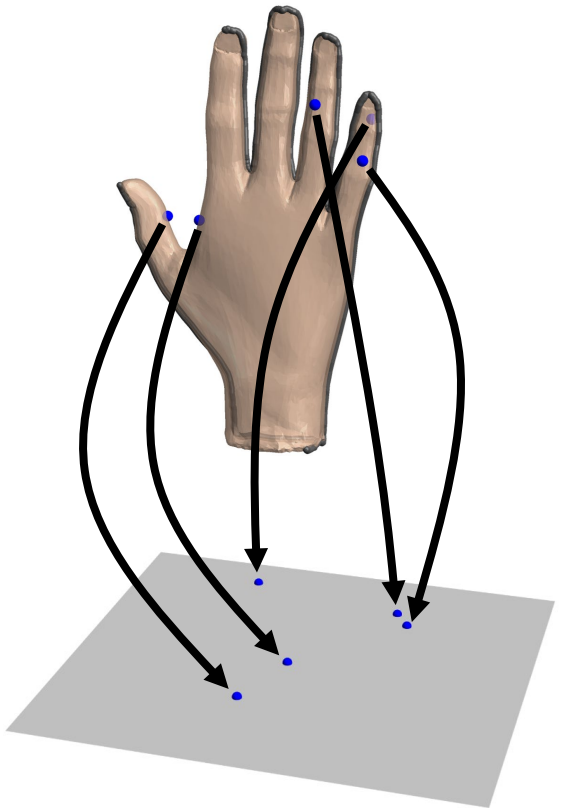
1

IsoTLC





Free-boundary mapping

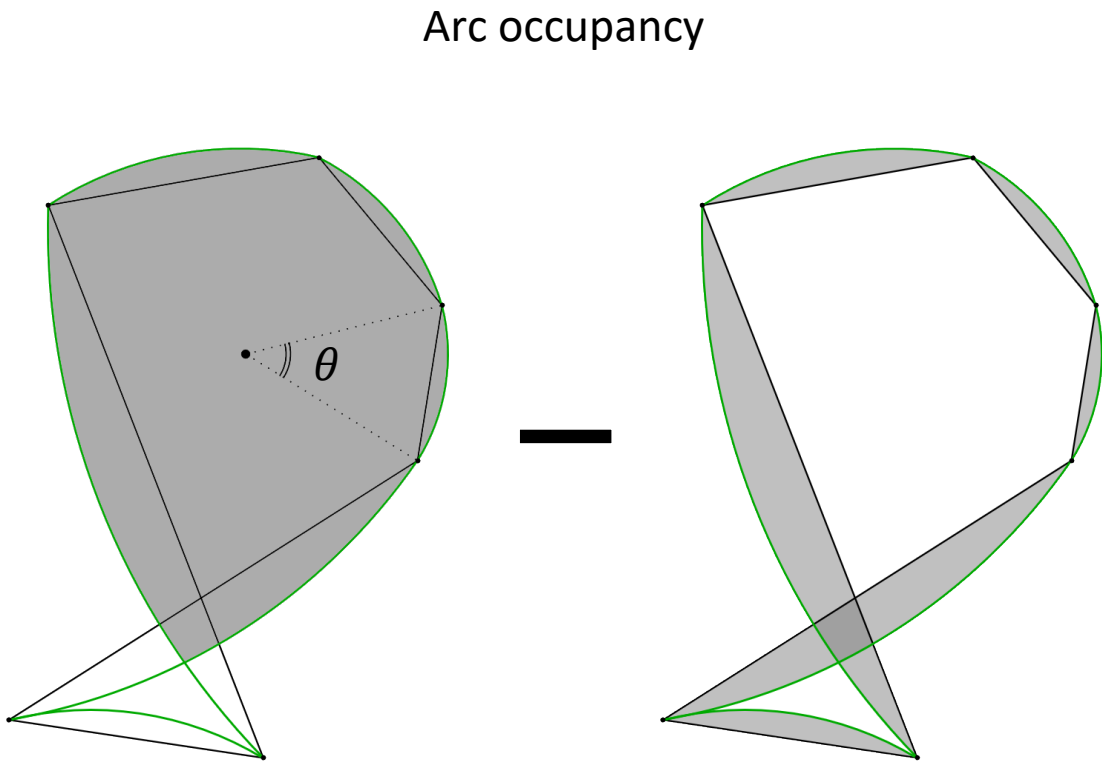


mapping with
positional constraints

Free-boundary mapping

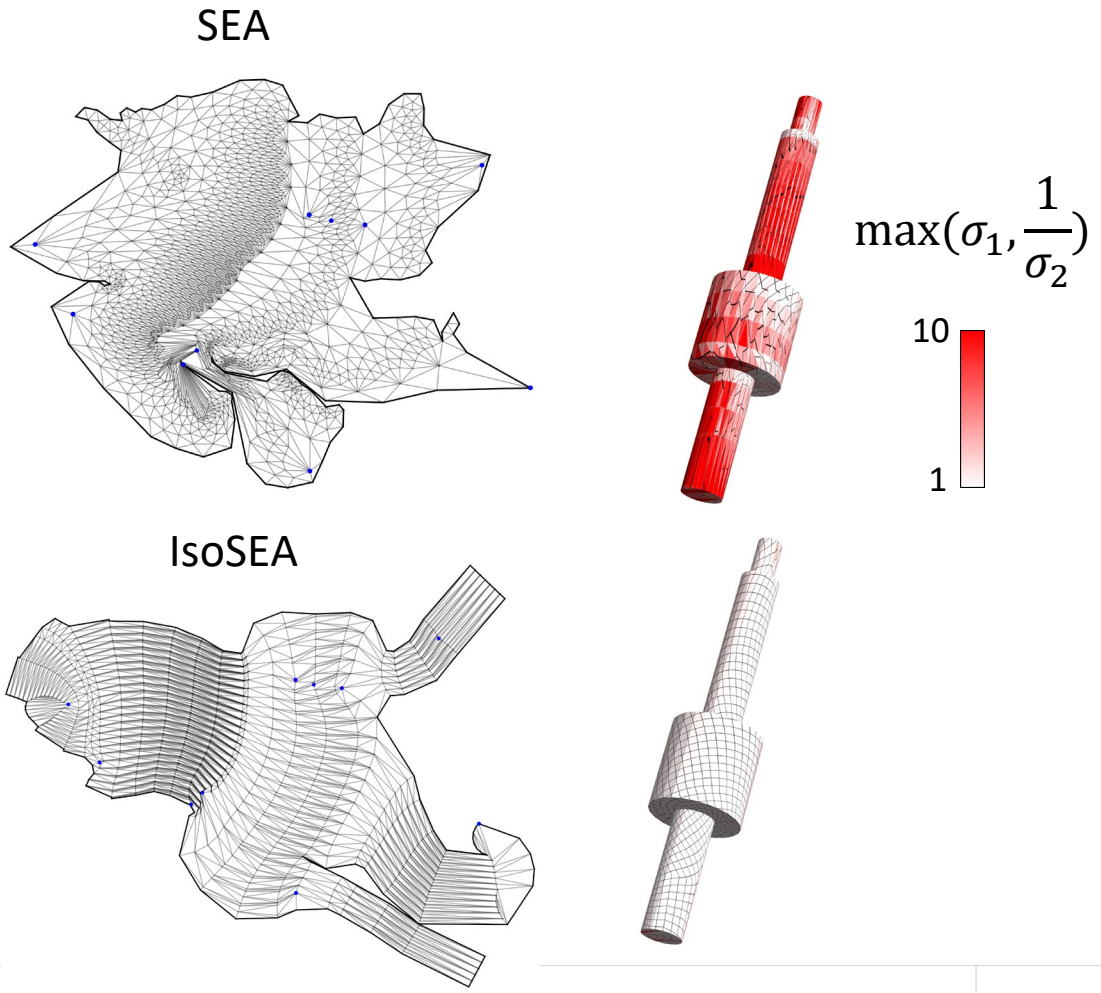
- Smooth excess area (SEA) [Du et al. 2021]
  - TLC minus arc occupancy

- Properties
  - Upper bound of the overlapping and inverted areas
  - Smoothly (almost everywhere) defined for injective and non-injective maps
  - Well-suited for gradient-based solvers
  - Global minima are locally injective with arbitrarily small overlapping (if they exist)
  - High success rate in recovering injectivity in 2D benchmark



Free-boundary mapping

- Isometric smooth excess area (**IsoSEA**)
  - **IsoTLC** minus arc occupancy
- Properties
  - Upper bound of the overlapping and inverted areas
  - Smoothly (almost everywhere) defined for injective and non-injective maps
  - Well-suited for gradient-based solvers
  - Global minima are locally injective with arbitrarily small overlapping (if they exist)
  - High success rate in recovering injectivity in 2D benchmark
  - Low area distortion



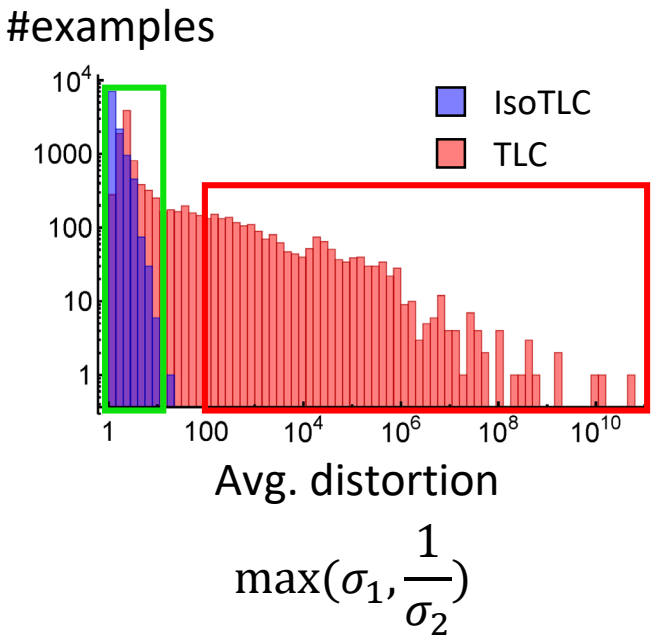
Optimization

- Quasi-Newton (IsoTLC, IsoSEA) and Projected Newton (IsoTLC)
- Computing injective maps
  - Map is injective
  - Reaches a max #iterations (e.g. 10 000)
- Lowering distortion
  - Energy converges
  - Reaches a max #iterations (e.g. 10 000)

Fixed-boundary mapping (2D, 3D)

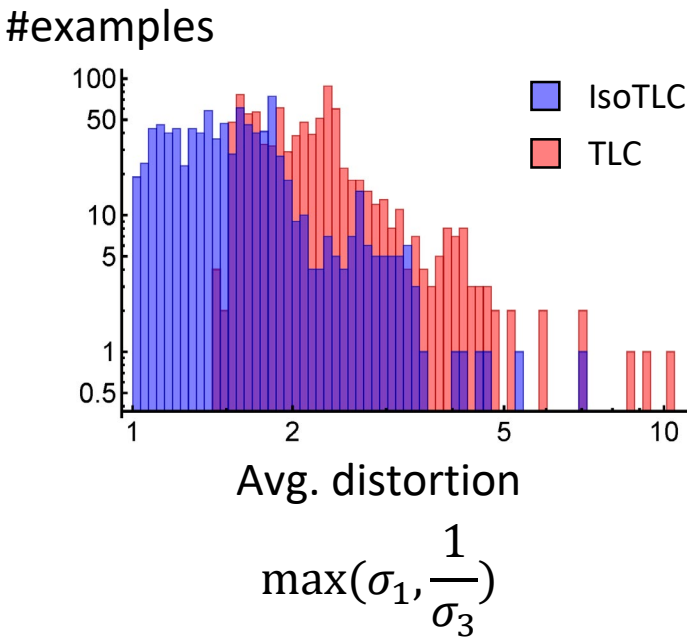
- Benchmark [Du et al. 2020]
  - 10743 triangle meshes
  - 904 tetrahedron meshes
- Parameter
  - $\alpha = 10^{-4}$
- Comparison with injectivity recovery methods
  - TLC [Du et al. 2020]
  - FFM [Garanzha et al. 2021]

2D



$$\max(\sigma_1, \frac{1}{\sigma_2})$$

3D

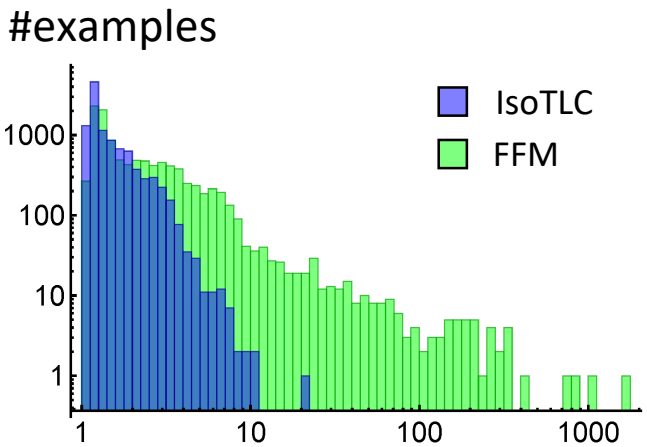


$$\max(\sigma_1, \frac{1}{\sigma_3})$$

Fixed-boundary mapping (2D, 3D)

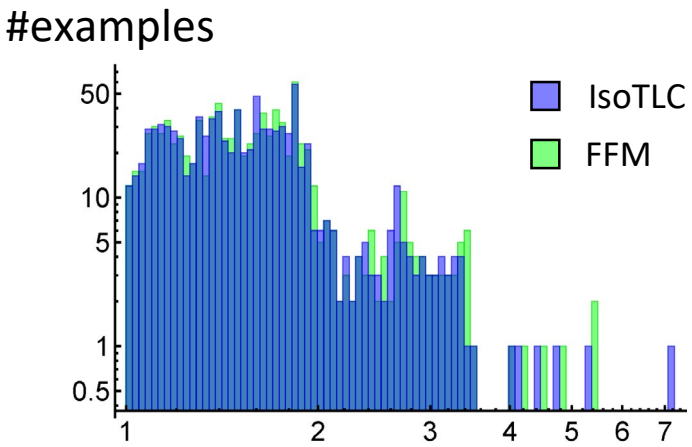
- Benchmark
  - 10743 triangle meshes
  - 904 tetrahedron meshes
- Parameter
  - $\alpha = 10^{-4}$
- Comparison with injectivity recovery methods
  - TLC [Du et al. 2020]
  - FFM [Garanzha et al. 2021]

2D



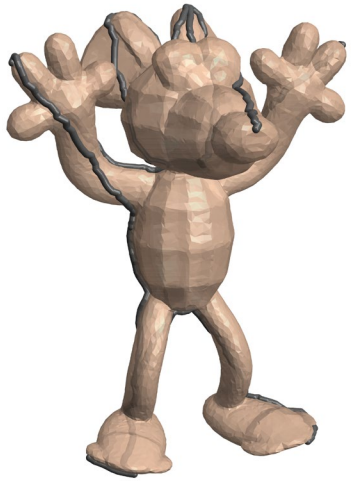
$$\max(\sigma_1, \frac{1}{\sigma_2})$$

3D

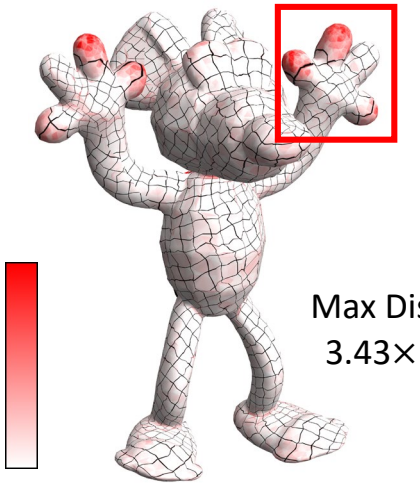


$$\max(\sigma_1, \frac{1}{\sigma_3})$$

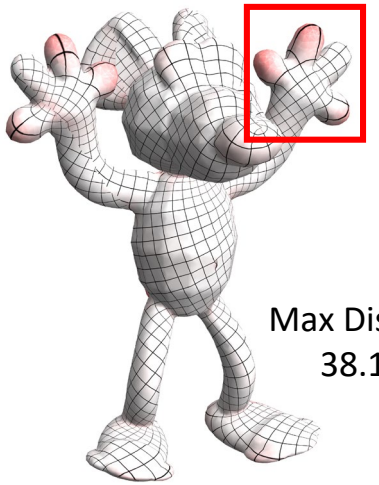
Fixed-boundary mapping



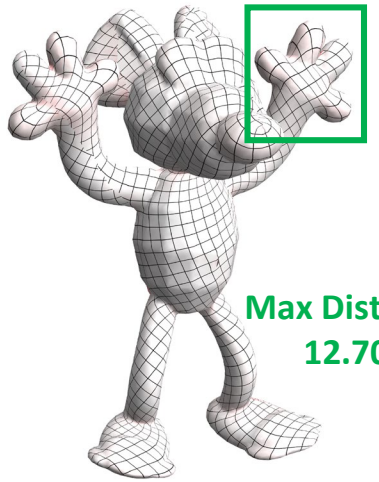
Source mesh



Max Distort:
3.43×10^5

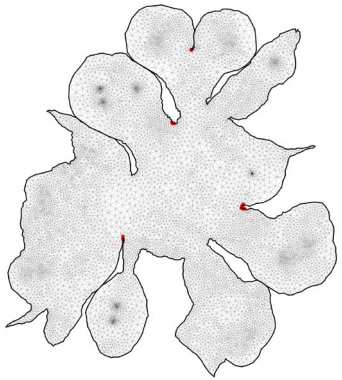


Max Distort:
38.15

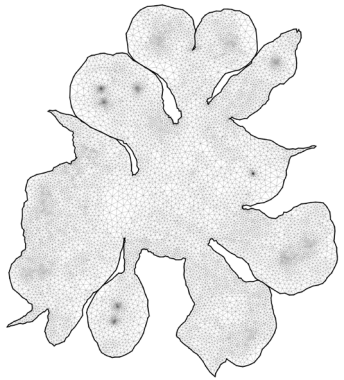


Max Distort:
12.70

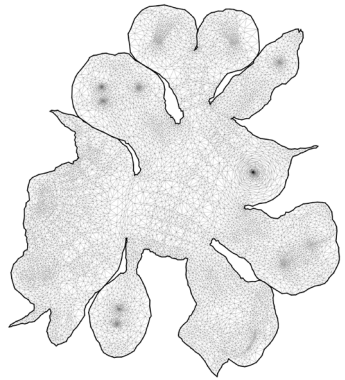
17
inverted
triangles



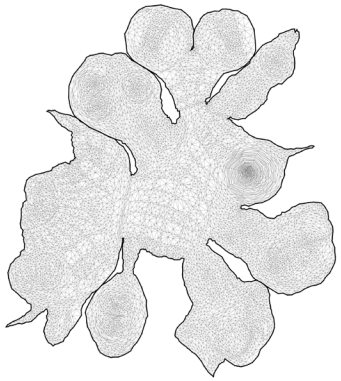
Initial map



TLC



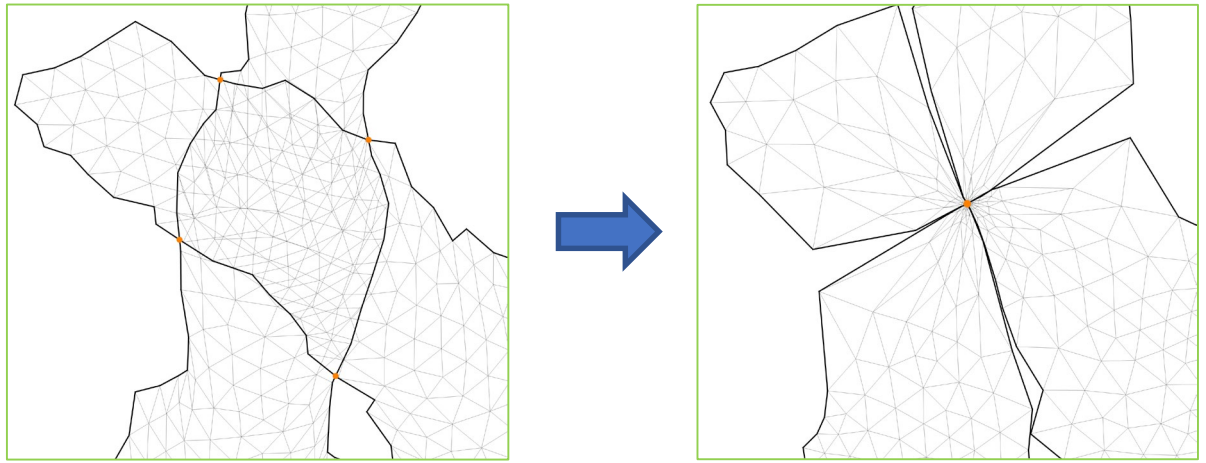
FFM



IsoTLC

Free-boundary mapping (2D)

- Benchmark [Du et al. 2021]
  - 1791 triangular meshes
  - Up to 20 constraints
- Parameters
  - $\alpha = 10^{-4}, \theta = 0.1$
- Success rate of recovering injective maps
  - SEA: 85 %
  - IsoSEA: 82 %



typical failure mode: crossing arm

Free-boundary mapping (2D)

- Benchmark [Du et al. 2021]
  - 1791 triangular mesh
  - Up to 20 constraints
- Parameters
  - $\alpha = 10^{-4}, \theta = 0.1$
- Success rate of recovering injective maps
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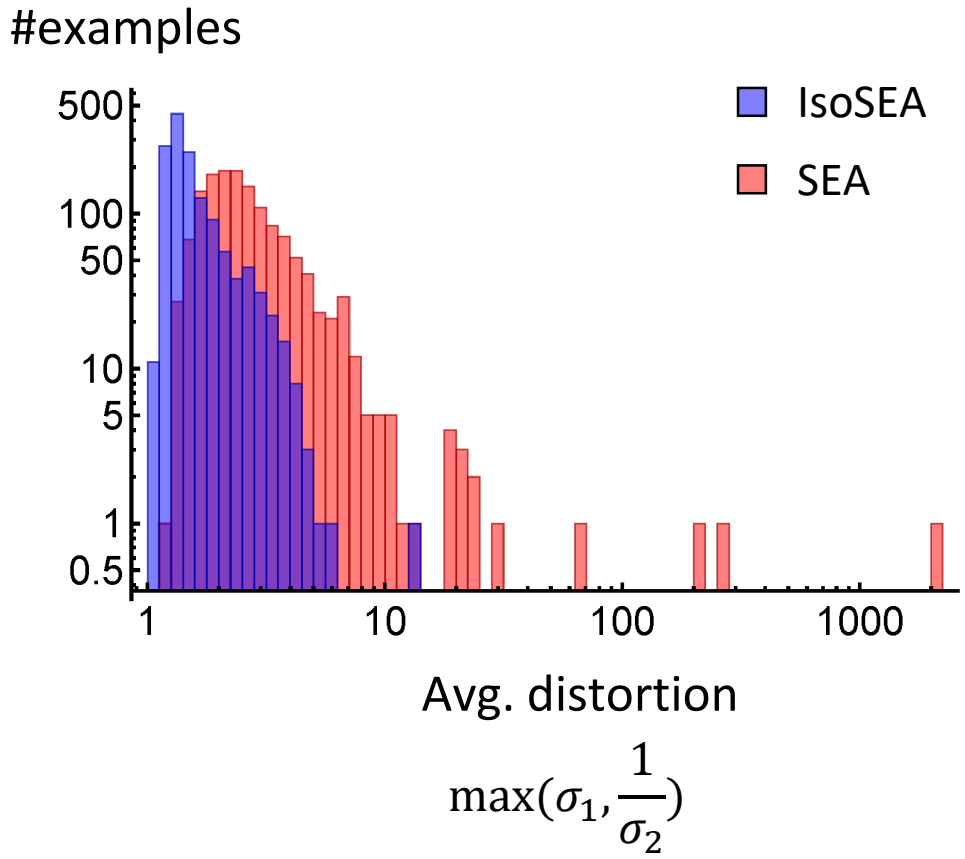
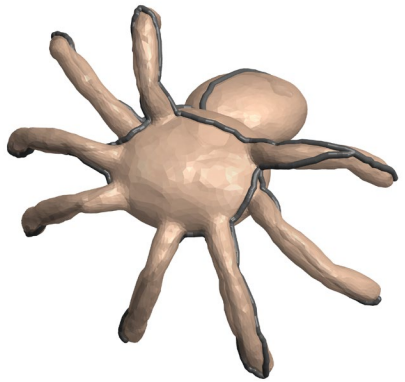
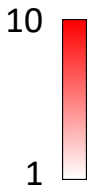
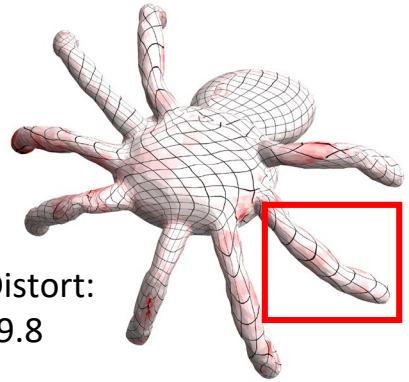


Free-boundary mapping

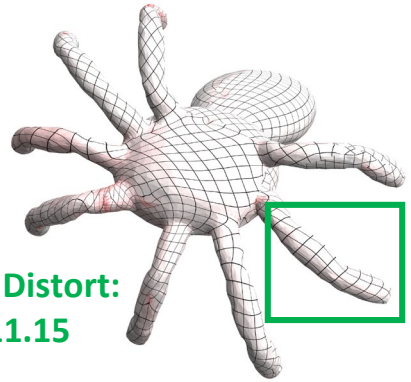


Source mesh





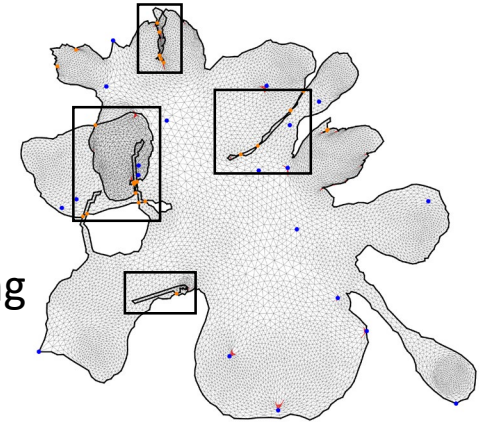
Max Distort:
139.8



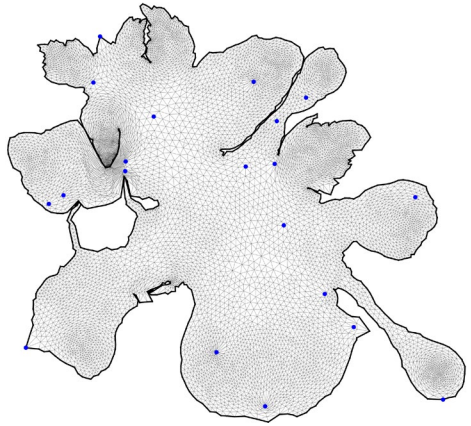
Max Distort:
11.15

63
inverted
triangles

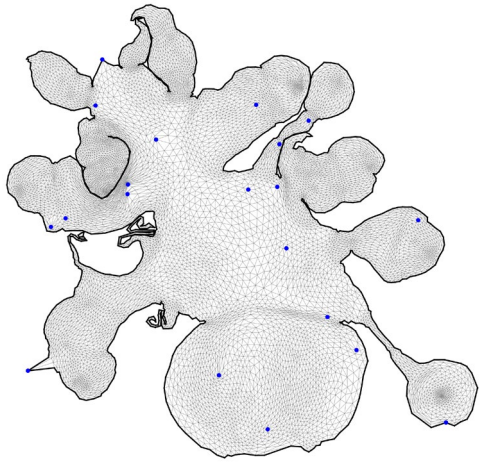
overlapping



Initial map



SEA

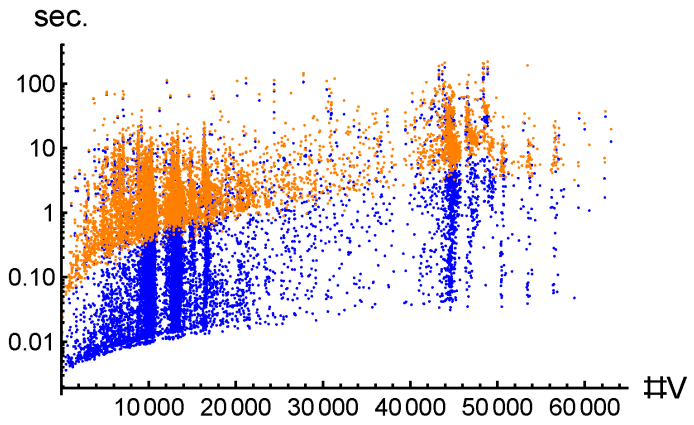


IsoSEA

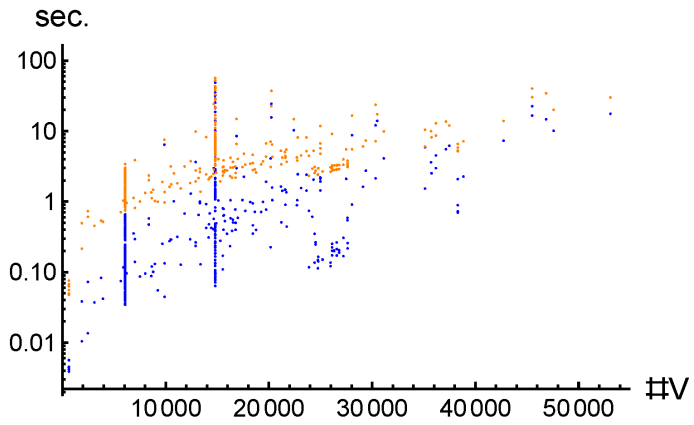
Performance

- Time(IsoSEA) >> Time(IsoTLC)
- Time(reduce distortion) >> Time(find injectivity)

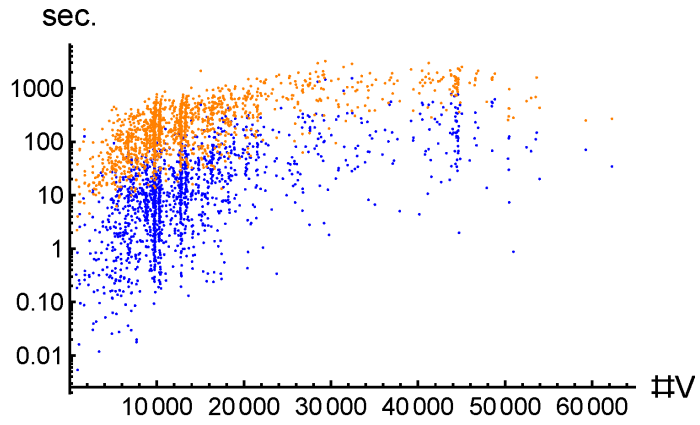
Fixed-boundary (2D)



Fixed-boundary (3D)



Free-boundary (2D)



● find injectivity ● find injectivity + reduce distortion

Summary

- Methods for computing injective, low distortion maps under constraints
  - Maintain high success rate
  - Reduce area distortion

- Future directions
  - Improve convergence rate
  - Free-boundary mapping in 3D
  - Adaptive parameters

Code and data